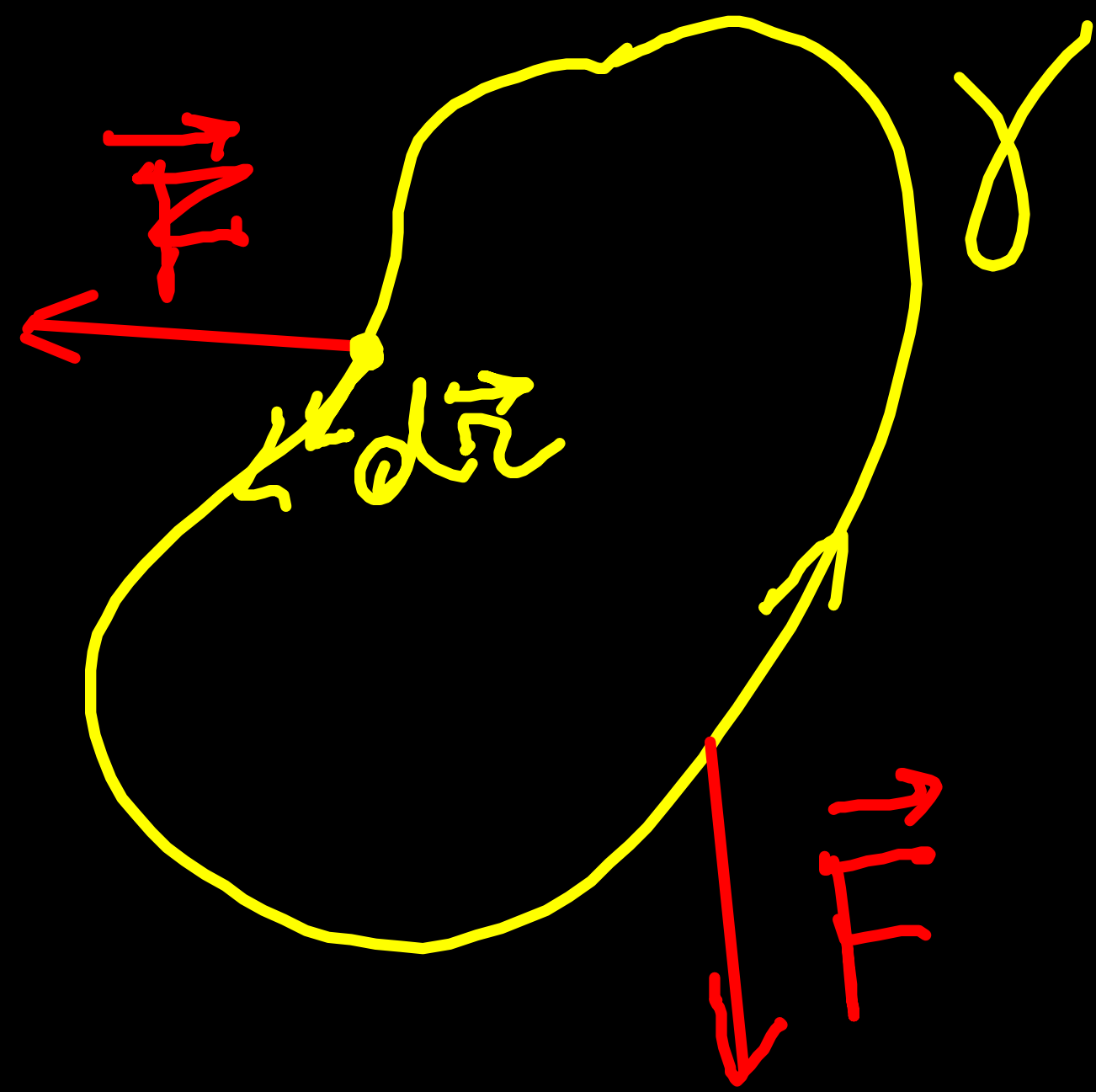


Def. Force conservative 3D

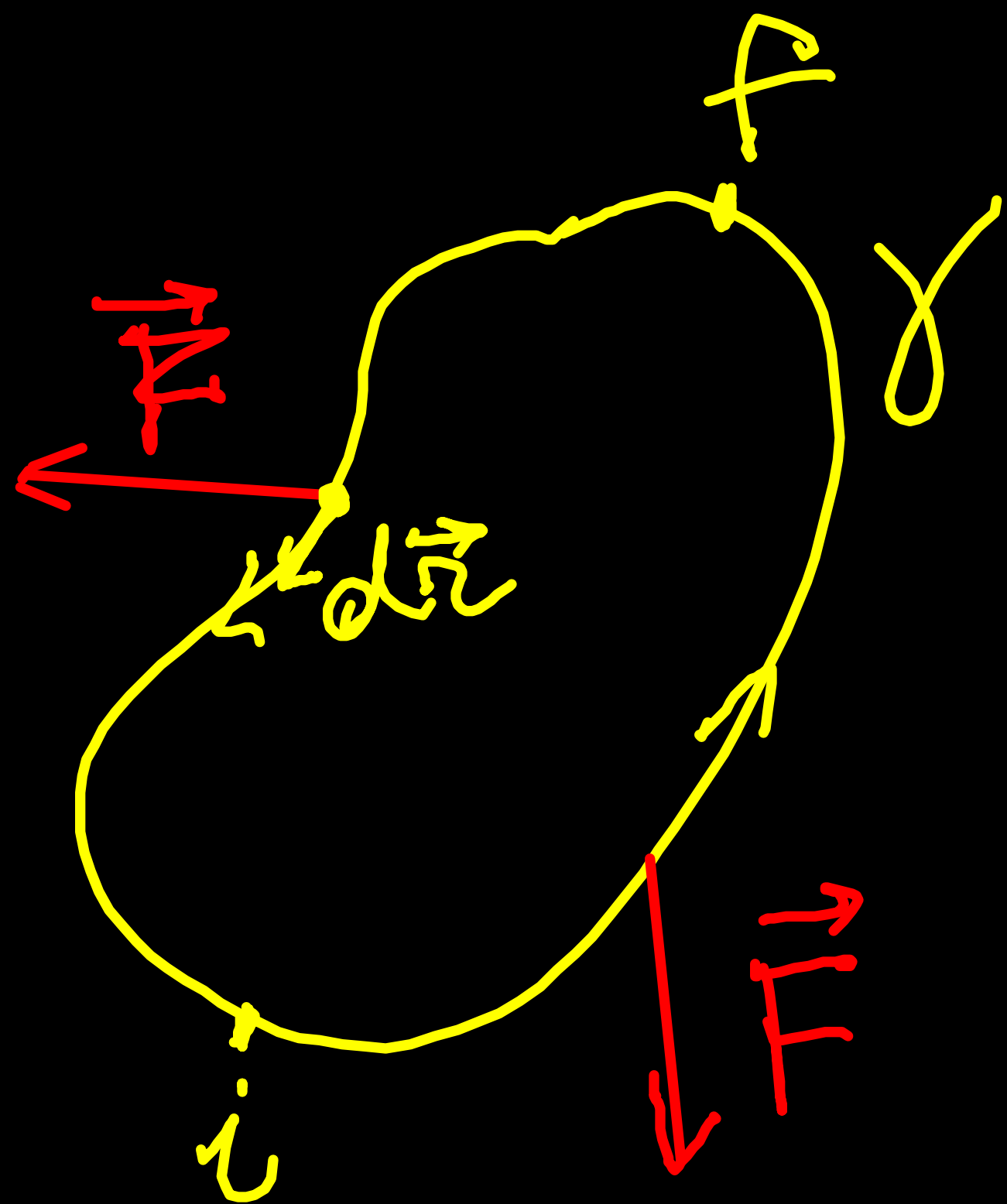


$$W_F = 0 \quad \forall \gamma$$

percorso chiuso

$$\oint \vec{F} \cdot d\vec{z} = 0$$

circulazione



$$\int_i F \cdot d\vec{z}$$

indipendente  
dal particolare  
percorso

Def. di energia potenziale  
associata a forze conservative 3D

$$U_f - U_i = - \int_i^f \vec{F} \cdot d\vec{z}$$

Esempi di Forze con. 3D

1) Forza peso  $\vec{F}_p = m\vec{g}$

$$\int_i^f \vec{F}_p \cdot d\vec{r} = -(mgy_f - mgy_i)$$

$$U_f - U_i = mgy_f - mgy_i$$

$$U(x, y, z) = mgy + \text{cost}$$

Altro esempio (tra poco)

2) Forza gravitazionale

$$\Delta U = U_f - U_i \equiv - \int_i^f \vec{F} \cdot d\vec{r}$$

$$-dU \equiv \vec{F} \cdot d\vec{r}$$

$$U = U(P)$$

Campo  
Scalare

Se scrivo  $U$  in coord. cart.

$$dU(x, y, z) \equiv \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$\vec{F} \cdot d\vec{r} \equiv F_x dx + F_y dy + F_z dz$$

È sempre

$$f(x, y) = y^2 \cdot x$$

$$\frac{\partial f}{\partial x} = y^2 \cdot \frac{\partial x}{\partial x} = y^2$$

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial y^2}{\partial y} = x \cdot 2y$$

gradiente

$$\left\{ \begin{aligned} F_x &= - \frac{\partial U}{\partial x} \\ F_y &= - \frac{\partial U}{\partial y} \\ F_z &= - \frac{\partial U}{\partial z} \end{aligned} \right.$$

$$\vec{F} = - \vec{\text{grad}} U$$
$$= - \nabla U$$

Oss.

1) Basta un campo scalare  
x det. il vettore forza  
conservativa



Oss. 2) esiste maniera più diretta  
x verificare che  $\oint \vec{F} \cdot d\vec{r} = 0 \quad \forall \gamma$  ?

Sì: mi aiuta l'operatore matematico

rotore

$$\text{rot } \vec{F} = \vec{\nabla} \times \vec{F}$$

- "Sonda" le proprietà rotazionali  
del campo vettoriale  $\vec{F}(P)$

- Se  $\vec{\nabla} \times \vec{F} = 0$  allora il campo  $\vec{F}$  conservativo o irrotazionale

In coord. cart.

$$\nabla \times \vec{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ F_x \\ F_y \\ F_z \end{pmatrix}$$

$$\oint \vec{F} \cdot d\vec{r} = 0 \iff \nabla \times \vec{F} = 0$$

$$\vec{F} = -\nabla U$$

$$\nabla \times \vec{F} = \nabla \times (-\nabla U) = \begin{bmatrix} \frac{\partial}{\partial y} \frac{\partial U}{\partial z} - \frac{\partial}{\partial z} \frac{\partial U}{\partial y} \end{bmatrix} \hat{i} + \dots = 0$$

$$= \left[ \frac{\partial}{\partial y} \frac{\partial U}{\partial z} - \frac{\partial}{\partial z} \frac{\partial U}{\partial y} \right] \hat{z} + \dots$$

$$= \frac{\partial}{\partial y} \frac{\partial U}{\partial z}$$

$$= 0$$

per lemma di Schwarz

Conserv. en. mecc. 3D

\* Teor. Em. Lav  $\Delta K = W_{TOT}$

Se agiscono solo forze conservative

$$\Rightarrow W_{TOT} = -\Delta U$$

$$\Delta K = -\Delta U$$

$$\Delta K = K_f - K_i$$

$$\Delta U = U_f - U_i$$

$$K_f + U_f = K_i + U_i$$

$$E = K + U \quad E_f = E_i$$

en. mecc. sist con.

Due note imp.

1) Sia sis. im. che non. imp.

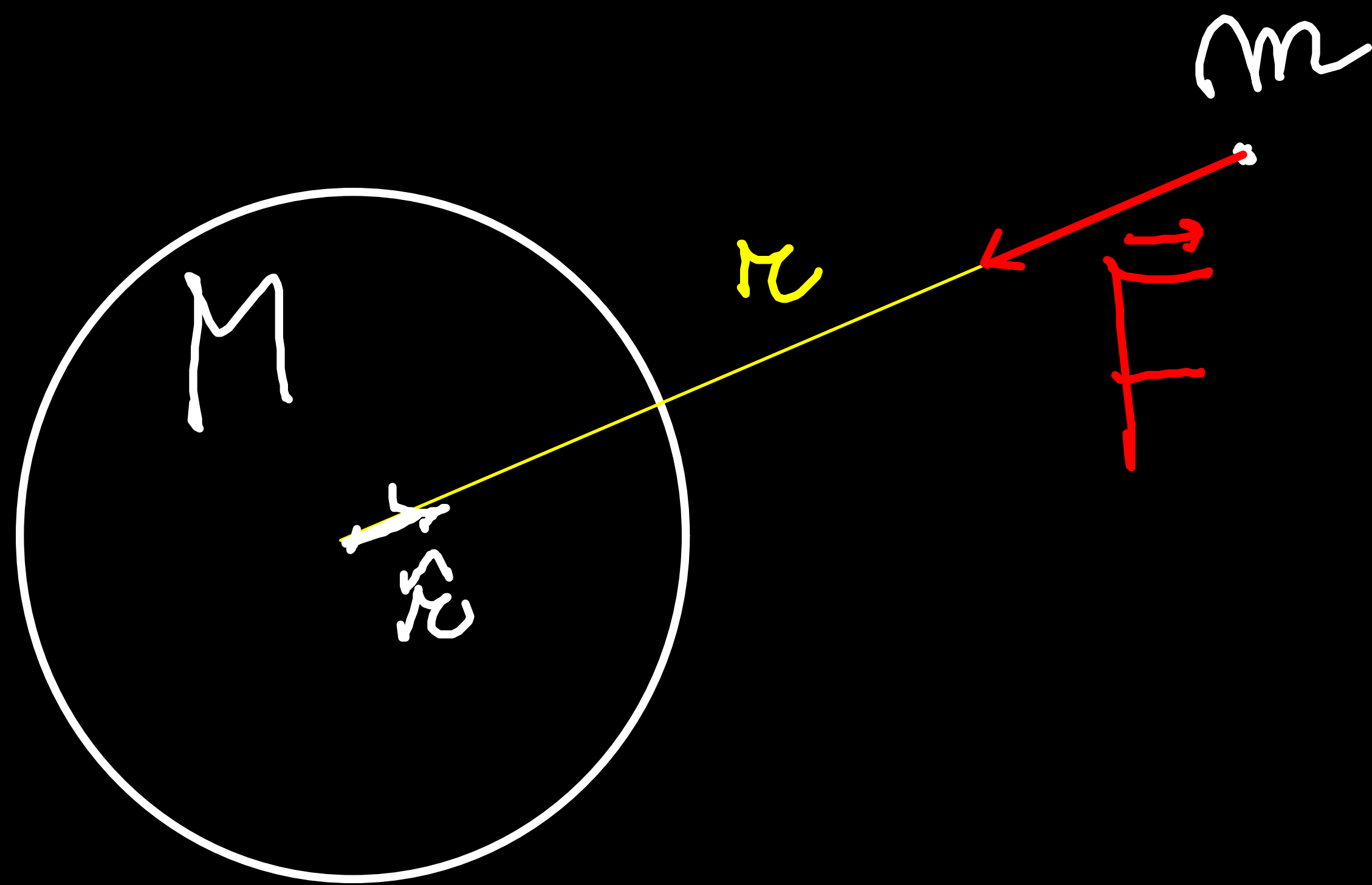
2) Se agiscono forze con lavoro nullo

$$\text{ok } W = 0$$

$$\text{and es. } \vec{F} \perp d\vec{r}$$

Altro esempio (tra poco)

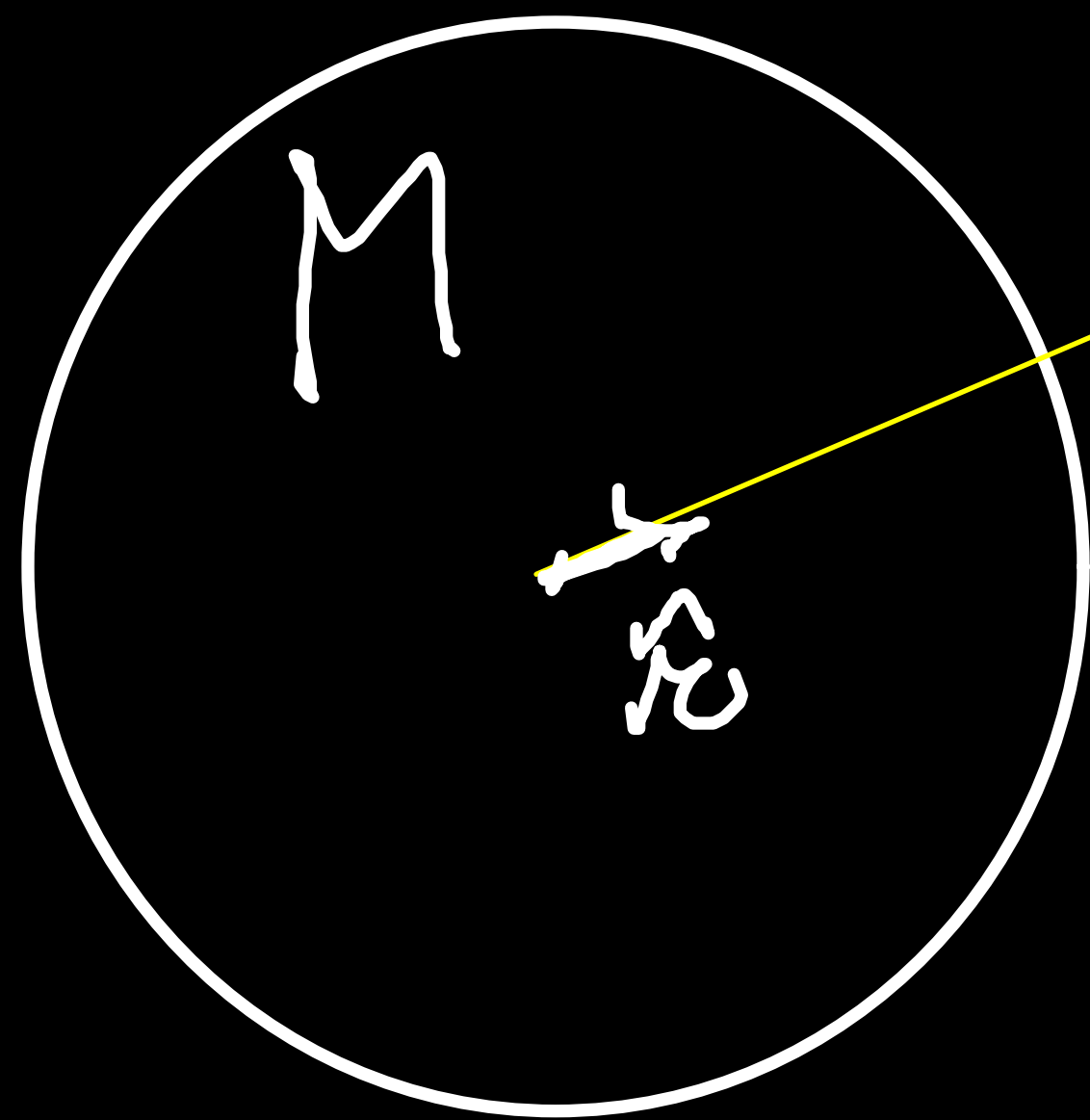
2) Forza gravitazionale



$$\vec{F} = -\frac{GMm}{r^2} \hat{r}$$

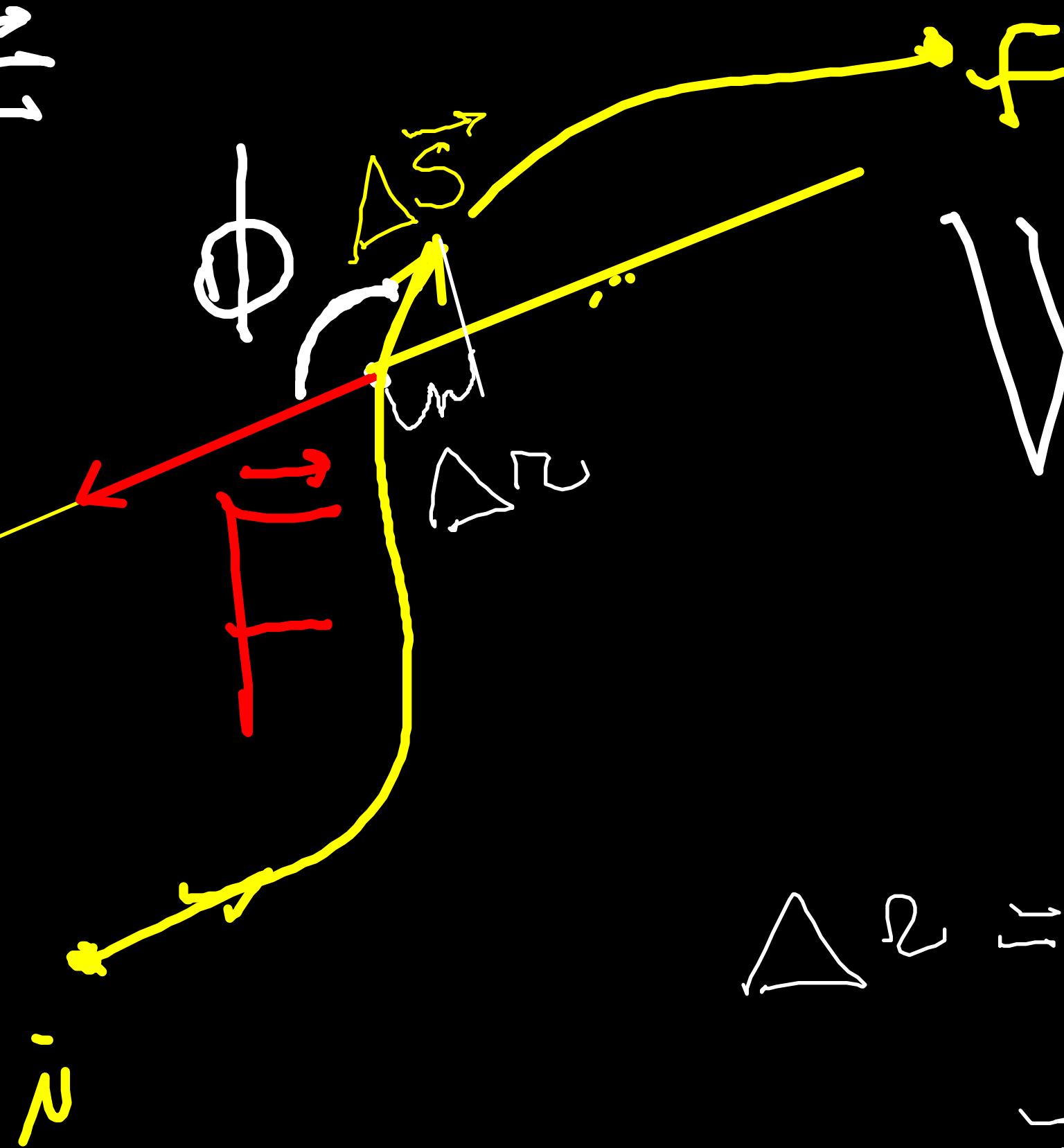
Nota:  $\vec{F}$  ha  
stessa direz.  
e verso opposto  $\hat{r}$

Curva  $U(P) \times \vec{H}$



$r$

$\vec{n}$



$$W = \int_i^f \vec{F} \cdot d\vec{S}$$

$$\Delta z = |\Delta \vec{S}| \cos(\pi - \phi)$$

$$\Rightarrow |\Delta \vec{S}| \cos \phi$$

$$\boxed{\vec{F} \cdot \Delta \vec{S}} \Rightarrow F |\Delta \vec{S}| \cos \phi = \boxed{-F \Delta z}$$

$\forall$  spostamento  $\Delta \vec{S}$

$W$  dipende solo  $\Delta z$

∇ traettoria

$$W = \int_{z_i}^{z_f} \vec{F} \cdot d\vec{s} = \int_{z_i}^{z_f} F dz =$$

$$= \int_{z_i}^{z_f} \frac{GMm}{r^2} dz = -GMm \int_{z_i}^{z_f} \frac{1}{r^2} dz$$

$$= -GMm \left[ -\frac{1}{r} \right]_{z_i}^{z_f} = GMm \left( \frac{1}{r_f} - \frac{1}{r_i} \right)$$



$$\Delta U \equiv -W$$

$$\Delta U = -GMm \left( \frac{1}{r_f} - \frac{1}{r_i} \right)$$

$$\rightarrow U(r) = -\frac{GMm}{r} + C$$

$$\Delta U \equiv -W$$

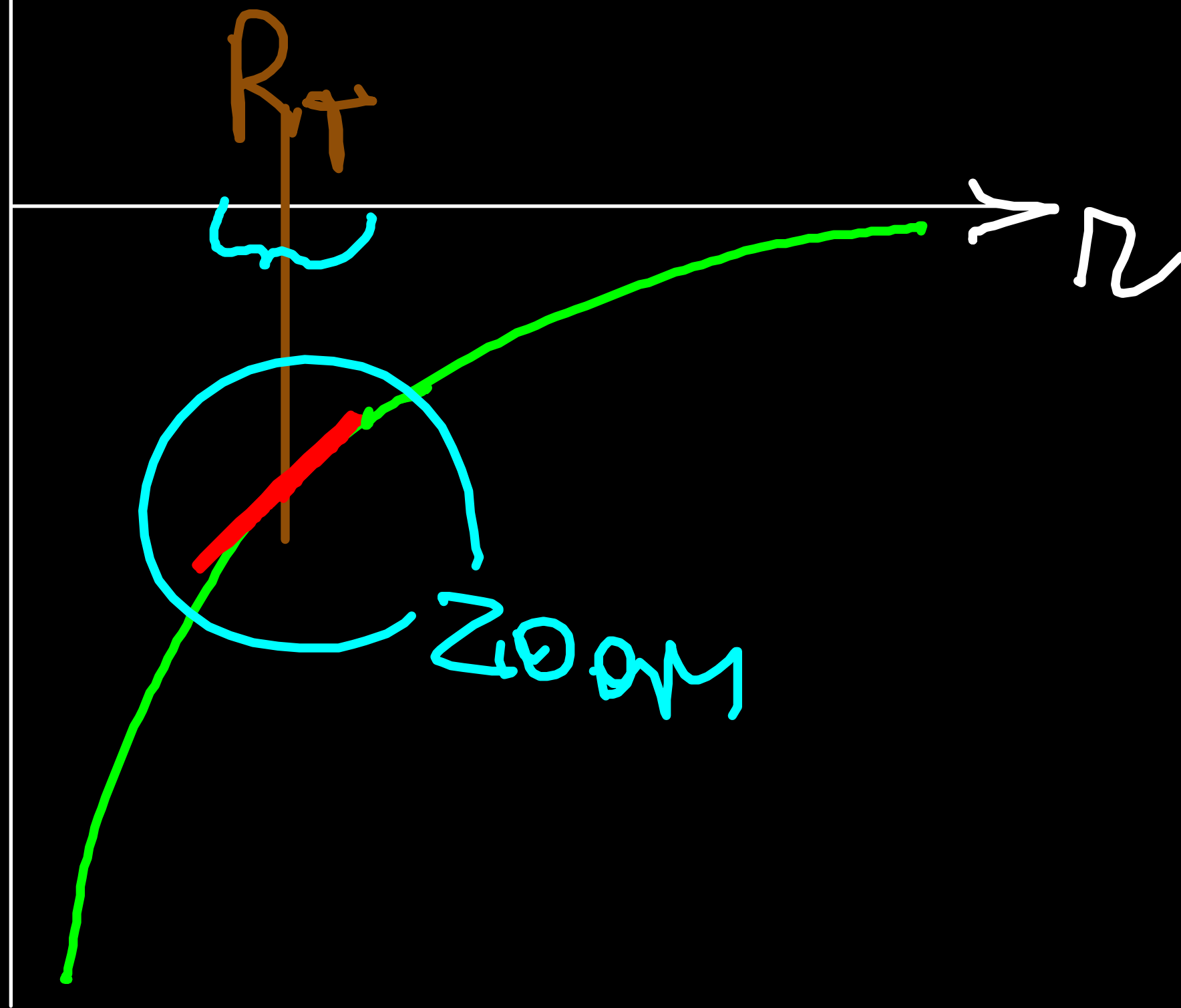
$$\Delta U \equiv -GMm \left( \frac{1}{r_f} - \frac{1}{r_i} \right)$$

$$\rightarrow U(r) = -\frac{GMm}{r} + C$$

Come posso scegliere C?

1) Situazione in cui si annulla  
 $F \xrightarrow{r \rightarrow \infty} 0$  a grandi dist  $U(r) \xrightarrow{r \rightarrow \infty} 0$   $C=0$

$U(r)$

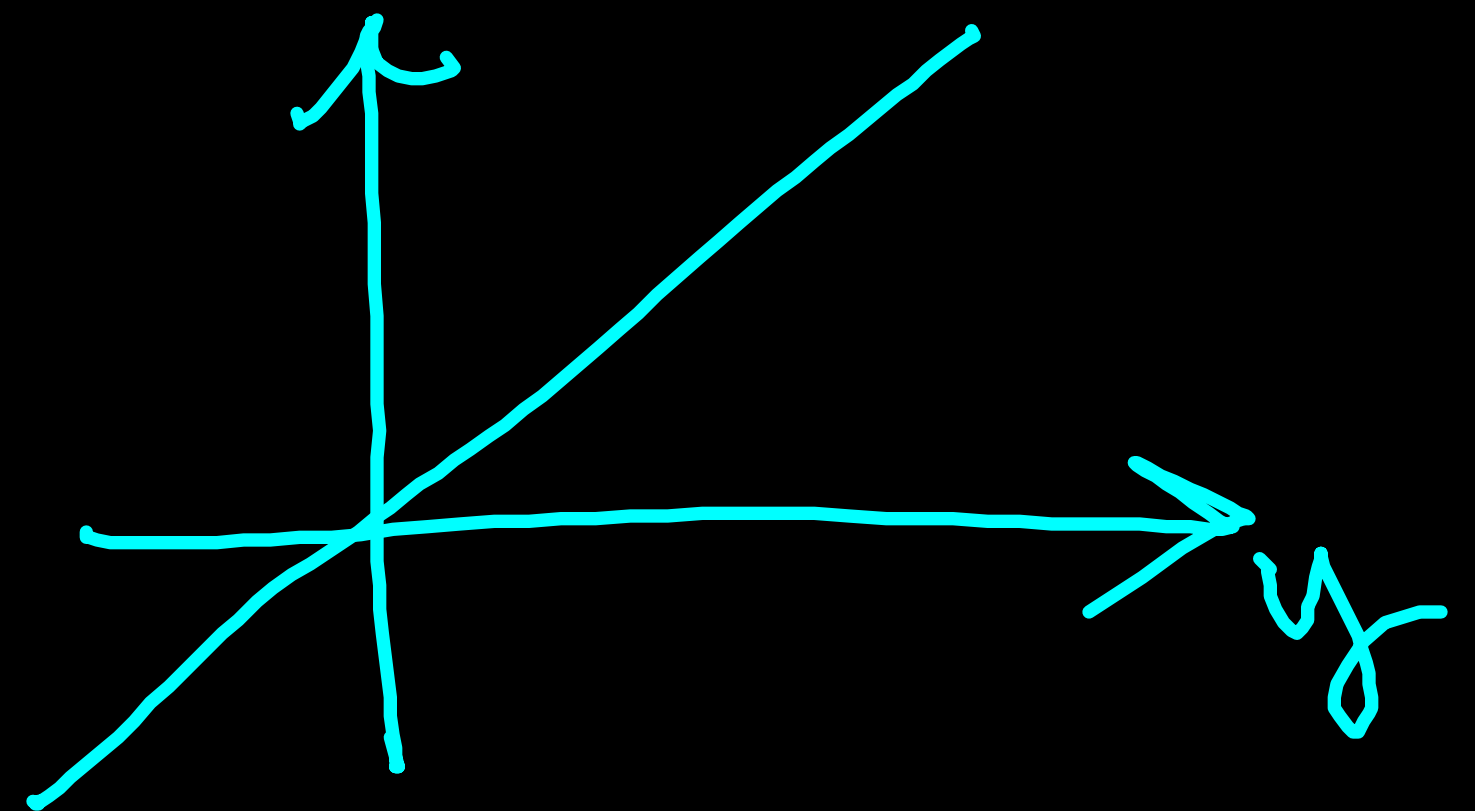


$$U(r) = -\frac{GMm}{r} + C$$

$\text{OK } \times r > R_T$

$C$  t.c.  $U(r=R_T) = 0$

$$U(y) = mgy$$



$$1) \quad U(y) \approx mgy \quad \rightarrow \quad \vec{F}_G = -\vec{\nabla} U$$

$$2) \quad U(z) = -\frac{GMm}{z} \quad \rightarrow \quad \vec{F}_P = -\vec{\nabla} U$$

$$\left| \vec{F}_G(z=R_H) \right| = \left| \vec{F}_P(y=0) \right|$$

$$\frac{GM}{R_H} = g$$

# Esempio 1

satellite in orb. att. Terra

$$U(r) = - \frac{G M_T m}{r}$$

$$\frac{1}{2} m v^2 + \left( - \frac{G M_T m}{r} \right) = E_{\text{cost.}}$$

orb. ellittica

quando è più lontano

$v$  aumenta e quindi  $v$  diminuisce

Esempio 2  
Se orbita è circolare

$U(r)$  e  $K$  rimangono cost.

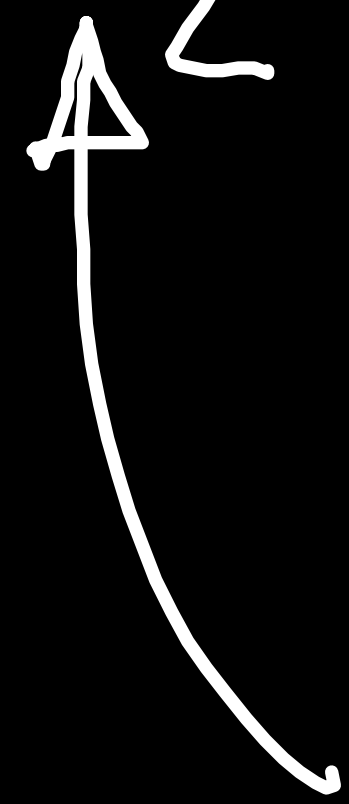
$$m \vec{a} = \vec{F}$$

$$m \frac{v^2}{r} = G \frac{m_1 m}{r^2} \quad / \quad \frac{1}{2} r$$

$$K = \frac{1}{2} m v^2 = \frac{1}{2} \frac{G m_1 m}{r} = -\frac{1}{2} U$$

$$E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U$$

$$E = -\frac{1}{2} \frac{G M m}{r}$$



energia di legame

Esempio 3

velocità di fuga

$K_i$   $v_i$  in modo che

orbita  $K_f \geq 0$   $r \rightarrow \infty$   
 $U_\infty = 0$

$$E_i = K_i + U_i \geq 0$$

$$\frac{1}{2} m v_i^2 - G \frac{m M}{r_i} \geq 0$$

$$v_{\text{fuga}} = \sqrt{\frac{2mMG}{r_i}}$$